

Appendix W7.14

Solution of State Equations

In this section, we consider the solution of state variable equations. This material is not necessary to understand the design of pole placement but will give a deeper insight into the method of state variables. It is instructive to consider first the unforced, or *homogenous*, system, which has the form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (\text{W7.1})$$

If the elements of $\mathbf{A}(t)$ are continuous functions of time, then the above equation has a *unique* solution for any initial state vector \mathbf{x}_0 . There is a useful representation for the solution of this equation in terms of a matrix, called the *transition matrix*. Let $\phi_i(t, t_0)$ be the solution to the special initial condition

$$\mathbf{x}(0) = \mathbf{e}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad \leftarrow i\text{th row} \quad (\text{W7.2})$$

If $\mathbf{x}(t_0)$ is the actual initial condition at t_0 , then we can express it in the decomposed form

$$\mathbf{x}(t_0) = \mathbf{x}_{01}\mathbf{e}_1 + \mathbf{x}_{02}\mathbf{e}_2 + \cdots + \mathbf{x}_{0n}\mathbf{e}_n. \quad (\text{W7.3})$$

Because Eq. (W7.1) is linear, the state $\mathbf{x}(t)$ also can be expressed as a sum of the solutions to the special initial condition ϕ_i , as

$$\mathbf{x}(t) = \mathbf{x}_{01}\phi_1(t, t_0) + \mathbf{x}_{02}\phi_2(t, t_0) + \cdots + \mathbf{x}_{0n}\phi_n(t, t_0), \quad (\text{W7.4})$$

or in matrix notation, as

$$\mathbf{x}(t) = \begin{bmatrix} \phi_1(t, t_0), & \phi_2(t, t_0), & \cdots, & \phi_n(t, t_0) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{01} \\ \mathbf{x}_{02} \\ \vdots \\ \mathbf{x}_{0n} \end{bmatrix}. \quad (\text{W7.5})$$

So we can define the *transition matrix*¹ to be

$$\Phi(t, t_0) = \begin{bmatrix} \phi_1(t, t_0), & \phi_2(t, t_0), & \cdots, & \phi_n(t, t_0) \end{bmatrix}, \quad (\text{W7.6})$$

¹This is also referred to as the *fundamental matrix* of the differential equation.

and write the solution as

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0), \quad (\text{W7.7})$$

where as the name implies, the transition matrix provides the transition between the state at time t_0 to the state at time t . Furthermore, from Eq. (W7.7), we have

$$\frac{d}{dt}[\mathbf{x}(t)] = \frac{d}{dt}[\Phi(t, t_0)]\mathbf{x}(t_0), \quad (\text{W7.8})$$

and from Eqs. (W7.1) and (W7.8), we have

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) = \mathbf{A}\Phi(t, t_0)\mathbf{x}(t_0). \quad (\text{W7.9})$$

Therefore

$$\frac{d}{dt}[\Phi(t, t_0)] = \mathbf{A}\Phi(t, t_0), \quad (\text{W7.10})$$

and also

$$\Phi(t, t) = \mathbf{I}. \quad (\text{W7.11})$$

The transition matrix can be shown to have many interesting properties. Among them are the following:

$$1. \quad \Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0); \quad (\text{W7.12})$$

$$2. \quad \Phi^{-1}(t, \tau) = \Phi(\tau, t); \quad (\text{W7.13})$$

$$3. \quad \frac{d}{dt}\Phi(t, \tau) = -\Phi(t, \tau)\mathbf{A}(\tau); \quad (\text{W7.14})$$

$$4. \quad \det \Phi(t, t_0) = e^{\int_{t_0}^t \text{trace} \mathbf{A}(\tau) d\tau}. \quad (\text{W7.15})$$

The second property implies that $\Phi(t, \tau)$ is always invertible. What this means is that the solution is always unique so, given a particular value of state at time τ , we can not only compute the future states from $\Phi(t, \tau)$ but also past values $\Phi^{-1}(t, \tau)$.

For the inhomogenous case, with a forcing function input $u(t)$, the equation is

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)u(t), \quad (\text{W7.16})$$

and the solution is

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0 + \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)u(\tau)d\tau. \quad (\text{W7.17})$$

We can verify this by substituting the supposed solution, Eq. (W7.17), into the differential equation, Eq. (W7.16), as

$$\frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}\Phi(t, t_0)\mathbf{x}_0 + \frac{d}{dt} \int_{t_0}^t \Phi(t, \tau)\mathbf{B}(\tau)u(\tau)d\tau. \quad (\text{W7.18})$$

The second term from calculus (using the Leibnitz formula) is

$$\frac{d}{dt} \int_{t_0}^t \Phi(t, \tau) \mathbf{B}(\tau) u(\tau) d\tau = \int_{t_0}^t \mathbf{A}(t) \Phi(t, \tau) \mathbf{B}(\tau) u(\tau) d\tau + \Phi(t, t) \mathbf{B}(t) u(t). \quad (\text{W7.19})$$

Using the basic relations, we have

$$\begin{aligned} \frac{d}{dt} \mathbf{x}(t) &= \mathbf{A}(t) \Phi(t, t_0) \mathbf{x}_0 + \mathbf{A}(t) \int_{t_0}^t \Phi(t, \tau) \mathbf{B}(\tau) u(\tau) d\tau \\ &\quad + \mathbf{B}(t) u(t), \end{aligned} \quad (\text{W7.20})$$

$$= \mathbf{A}(t) \mathbf{x}(t) + \mathbf{B}(t) u(t), \quad (\text{W7.21})$$

which shows that the proposed solution satisfies the system equation. For the time-invariant case

$$\Phi(t, t_0) = e^{\mathbf{A}(t-t_0)} = \Phi(t - t_0), \quad (\text{W7.22})$$

where

$$e^{\mathbf{A}t} = \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \cdots \right) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}, \quad (\text{W7.23})$$

is an invertible $n \times n$ exponential matrix, and by letting $t = 0$, we see that

$$e^{\mathbf{0}} = \mathbf{I}. \quad (\text{W7.24})$$

The state solution is now

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_0)} \mathbf{x}_0 + \int_{t_0}^t \mathbf{e}^{\mathbf{A}(t-\tau)} \mathbf{B} u(\tau) d\tau, \quad (\text{W7.25})$$

and

$$y(t) = \mathbf{C} \mathbf{x}(t) = \mathbf{C} \mathbf{e}^{\mathbf{A}(t-t_0)} \mathbf{x}_0 + \mathbf{C} \int_{t_0}^t \mathbf{e}^{\mathbf{A}(t-\tau)} \mathbf{B} u(\tau) d\tau + D u(t). \quad (\text{W7.26})$$

Suppose $\mathbf{x}(t_0) = \mathbf{x}_0 \equiv \mathbf{0}$, then the output is given by the convolution integral

$$y(t) = \int_{t_0}^t \mathbf{h}(t - \tau) \mathbf{B} u(\tau) d\tau, \quad (\text{W7.27})$$

where $h(t)$ is the *impulse response*. In terms of the state variables matrices,

$$h(t) = \mathbf{C} e^{\mathbf{A}t} \mathbf{B} + D \delta(t). \quad (\text{W7.28})$$

While there is no uniformly best way to compute the transition matrix, there are several methods that can be used to compute accurate approximations to it (See Moler, 2003; Franklin, Powell, and Workman,

1998). Three of these methods are matrix exponential series, inverse Laplace transform, and diagonalization of the system matrix. In the first technique, we use Eq. (W7.23):

$$e^{At} \triangleq \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots, \quad (\text{W7.29})$$

and the series should be computed in a reliable fashion. For the second method, we notice if we define

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}, \quad (\text{W7.30})$$

then we can compute $\Phi(s)$ from the \mathbf{A} matrix and matrix algebra. Given this matrix, we can use the inverse Laplace Transform to compute

$$\Phi(t) = \mathcal{L}^{-1} \{ \Phi(s) \}, \quad (\text{W7.31})$$

$$= \mathcal{L}^{-1} \left\{ (s\mathbf{I} - \mathbf{A})^{-1} \right\}. \quad (\text{W7.32})$$

The last method we mentioned operates on the system matrix. If the system matrix can be diagonalized, that is, if we can find a transformation matrix \mathbf{T} so that

$$\Lambda = \mathbf{T}^{-1} \mathbf{A} \mathbf{T}, \quad (\text{W7.33})$$

where Λ is reduced to the similar but diagonal matrix

$$\Lambda = \text{diag} \{ \lambda_1, \lambda_2, \dots, \lambda_n \}, \quad (\text{W7.34})$$

then from the series, Eq. (W7.23), we need only compute scalar exponentials, since

$$e^{At} = \mathbf{T}^{-1} \text{diag} \{ e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t} \} \mathbf{T}. \quad (\text{W7.35})$$