

Appendix W3.2.3

△ W3.2.3 Mason's Rule and the Signal-Flow Graph

A compact alternative notation to the block diagram is given by the **signal-flow graph** introduced by S. J. Mason (1953, 1956). As with the block diagram, the signal-flow graph offers a visual tool for representing the causal relationships between the components of the system. The method consists of characterizing the system by a network of directed branches and associated gains (transfer functions) connected at nodes. Several block diagrams and their corresponding signal-flow graphs are shown in Fig. W3.1. The two ways of depicting a system are equivalent, and you can use either diagram to apply Mason's rule (to be defined shortly).

Signal-flow graph

In a signal-flow graph, the internal signals in the diagram, such as the common input to several blocks or the output of a summing junction, are called **nodes**. The system input point and the system output point are also nodes; the input node has outgoing branches only, and the output node has incoming branches only. Mason defined a **path** through a block diagram as a sequence of connected blocks, the route passing from one node to another *in the direction of signal flow of the blocks* without including any block more than once. A **forward path** is a path from the input to output such that no node is included more than once. If the nodes are numbered in a convenient order, a forward path can be identified by the numbers that are included. Any closed path that returns to its starting node without passing through any node more than once is a **loop**, and a path that leads from a given variable back to the same variable is a **loop path**. The **path gain** is the product of component gains (transfer functions) making up the path. Similarly, the **loop gain** is the path gain associated with a loop—that is, the product of gains in a loop. If two paths have a common component, they are said to touch. Notice particularly in this connection that the input and the output of a summing junction are not the same and that the summing junction is a one-way device from its inputs to its output.

Mason's rule relates the graph to the algebra of the simultaneous equations it represents.¹ Consider Fig. W3.1c, where the signal at each node has been given a name and the gains are marked. Then the block diagram (or the signal-flow graph) represents the system of equations:

¹The derivation is based on Cramer's rule for solving linear equations by determinants and is described in Mason's papers.

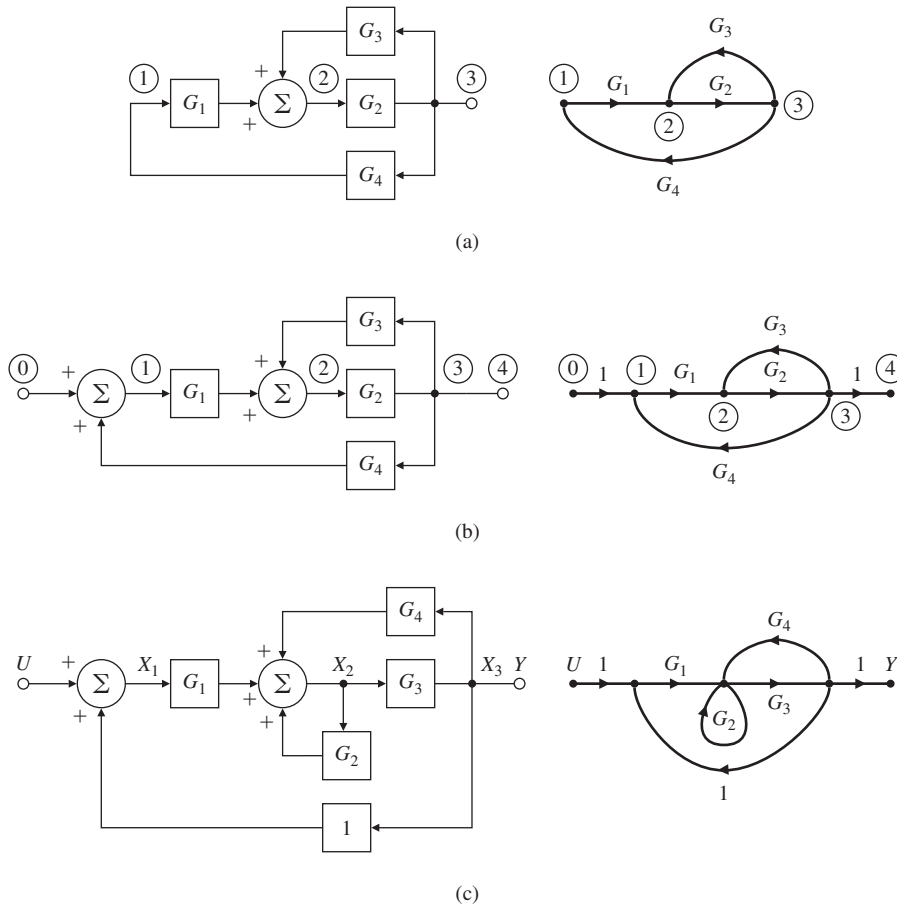


Figure W3.1
Block diagrams and corresponding signal-flow graphs

$$\begin{aligned}
 X_1(s) &= X_3(s) + U(s), \\
 X_2(s) &= G_1(s)X_1(s) + G_2(s)X_2(s) + G_4(s)X_3(s), \\
 Y(s) &= 1X_3(s).
 \end{aligned}$$

Mason's rule

Mason's rule states that the input-output transfer function associated with a signal-flow graph is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\Delta} \sum_i G_i \Delta_i,$$

where

G_i = path gain of the i th forward path,

Δ = the system determinant

$$= 1 - \sum (\text{all individual loop gains}) + \sum (\text{gain products of all possible two loops that do not touch}) - \sum (\text{gain products of all possible three loops that do not touch}) + \dots,$$

Δ_i = i th forward path determinant

= value of Δ for that part of the block diagram that does *not* touch the i th forward path.

We will now illustrate the use of Mason's rule with some examples.

EXAMPLE W3.1

Mason's Rule in a Simple System

Find the transfer function for the block diagram in Fig. W3.2.

Solution. From the block diagram shown in Fig. W3.2, we have

Forward Path	Path Gain
1236	$G_1 = 1 \left(\frac{1}{s} \right) (b_1)(1)$
12346	$G_2 = 1 \left(\frac{1}{s} \right) \left(\frac{1}{s} \right) (b_2)(1)$
123456	$G_3 = 1 \left(\frac{1}{s} \right) \left(\frac{1}{s} \right) \left(\frac{1}{s} \right) (b_3)(1)$
Loop Path Gain	
232	$l_1 = -a_1/s$
2342	$l_2 = -a_2/s^2$
23452	$l_3 = -a_3/s^3$

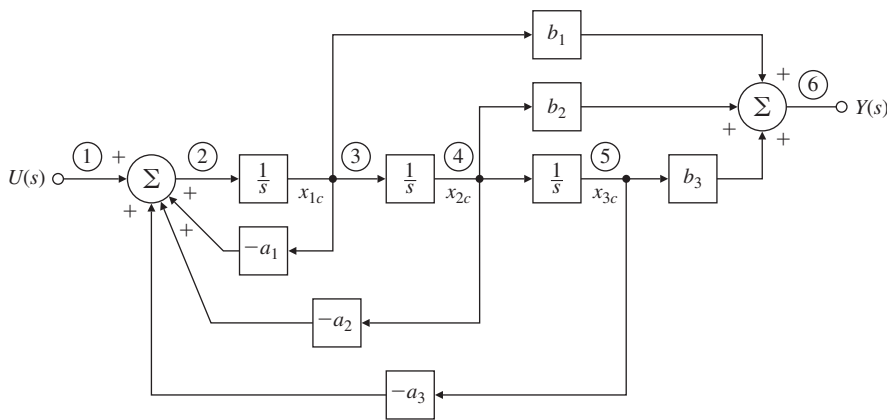


Figure W3.2

Block diagram for Example W3.1

and the determinants are

$$\Delta = 1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3} \right) + 0,$$

$$\Delta_1 = 1 - 0,$$

$$\Delta_2 = 1 - 0,$$

$$\Delta_3 = 1 - 0.$$

Applying Mason's rule, we find the transfer function to be

$$\begin{aligned} G(s) &= \frac{Y(s)}{U(s)} = \frac{(b_1/s) + (b_2/s^2) + (b_3/s^3)}{1 + (a_1/s) + (a_2/s^2) + (a_3/s^3)} \\ &= \frac{b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}. \end{aligned}$$

Mason's rule is particularly useful for more complex systems where there are several loops, some of which do not sum into the same point.

EXAMPLE W3.2

Mason's Rule in a Complex System

Find the transfer function for the system shown in Fig. W3.3.

Solution. From the block diagram, we find that

Forward Path	Path Gain
12456	$G_1 = H_1H_2H_3$
1236	$G_2 = H_4$
<i>Loop Path Gain</i>	
242	$l_1 = H_1H_5$ (does not touch l_3)
454	$l_2 = H_2H_6$
565	$l_3 = H_3H_7$ (does not touch l_1)
236542	$l_4 = H_4H_7H_6H_5$

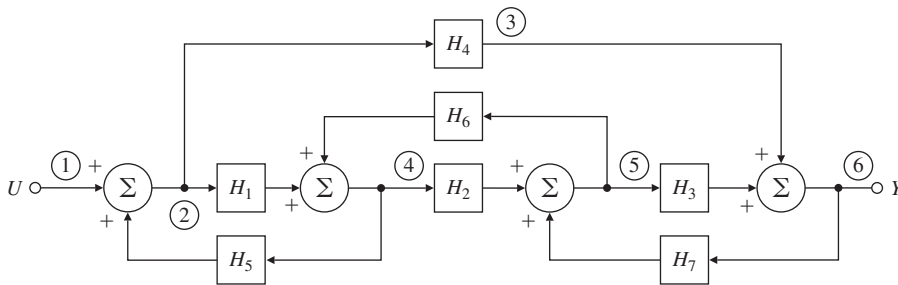


Figure W3.3

Block diagram for Example W3.2

and the determinants are

$$\Delta = 1 - (H_1H_5 + H_2H_6 + H_3H_7 + H_4H_7H_6H_5) + (H_1H_5H_3H_7),$$

$$\Delta_1 = 1 - 0,$$

$$\Delta_2 = 1 - H_2H_6.$$

Therefore,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{H_1H_2H_3 + H_4 - H_4H_2H_6}{1 - H_1H_5 - H_2H_6 - H_3H_7 - H_4H_7H_6H_5 + H_1H_5H_3H_7}.$$

Mason's rule is useful for solving relatively complicated block diagrams by hand. It yields the solution in the sense that it provides an explicit input-output relationship for the system represented by the diagram. The advantage compared with path-by-path block-diagram reduction is that it is systematic and algorithmic rather than problem dependent. Matlab and other control systems computer-aided software allow you to specify a system in terms of individual blocks in an overall system, and the software algorithms perform the required block-diagram reduction; therefore, Mason's rule is less important today than in the past. However, there are some derivations that rely on the concepts embodied by the rule, so it still has a role in the control designer's toolbox.